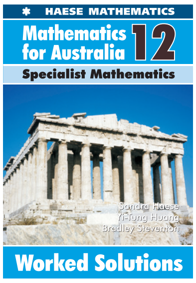


ERRATA

MATHEMATICS FOR AUSTRALIA 12

Specialist Mathematics

Worked Solutions

First edition - 2017

The following errata were made on 05/May/2020

page 151 **EXERCISE 4B** Question **10 b**, sixth line should read:

10 **b** $\therefore \; 8zz^* = 288$

page 156 **EXERCISE 4C.1** Question **2**, should read:

2 $z = 0 = 0 + 0i$ cannot be written in polar form. The vector representing z has length zero, and an argument is not defined (no angle can be formed with the positive x-axis).

page 178 **EXERCISE 4E** Question **13 c**, should read:

13 **c** **i** $z = 2\,\text{cis}\left(\frac{\pi}{7}\right)$

$$z \times z^2 \times z^3 \times \ldots. \times z^k = 2^{\frac{k(k+1)}{2}}\,\text{cis}\left[\frac{k(k+1)\pi}{14}\right]$$

$$= 2^{\frac{k(k+1)}{2}}\left[\cos\left(\frac{k(k+1)\pi}{14}\right) + i\sin\left(\frac{k(k+1)\pi}{14}\right)\right]$$

which is real when $\sin\left(\frac{k(k+1)\pi}{14}\right) = 0$

$$\therefore \; \frac{k(k+1)\pi}{14} = n\pi, \quad n \in \mathbb{Z}$$

$$\therefore \; k(k+1) = 14n$$

which has smallest integer solution $k = 6, \; n = 3$

$$\therefore \; \left|z^1 \times z^2 \times z^3 \times \ldots. \times z^6\right| = 2^{\frac{6(7)}{2}}$$

$$= 2^{21}$$

ii $z \times z^2 \times z^3 \times \ldots. \times z^k = 2^{\frac{k(k+1)}{2}}\left[\cos\left(\frac{k(k+1)\pi}{14}\right) + i\sin\left(\frac{k(k+1)\pi}{14}\right)\right]$

which is purely imaginary when $\cos\left(\frac{k(k+1)\pi}{14}\right) = 0$

$$\therefore \; \frac{k(k+1)\pi}{14} = (2n-1)\frac{\pi}{2}, \quad n \in \mathbb{Z}^+$$

$$\therefore \; \frac{k(k+1)}{14} = \frac{2n-1}{2}$$

$$\therefore \; k(k+1) = 7(2n-1) \quad \ldots. \; (*)$$

But $k(k+1)$ is even for all $k \in \mathbb{Z}^+$ and $7(2n-1)$ is odd for all $n \in \mathbb{Z}^+$.

So, $(*)$ is never true.

$\therefore \; z^1 \times z^2 \times z^3 \times \ldots. \times z^k$ is never purely imaginary.

page 188 **EXERCISE 4F.2** Question **4 b ii**, should read:

4 b ii $1+w+w^2+....+w^{n-1}$ is a geometric series with $t_1 = 1$ and $r = w = \text{cis}\left(\frac{2\pi}{n}\right)$.

$\therefore$ it has sum $S_n = \dfrac{t_1(r^n-1)}{r-1}$

$$= \frac{1(w^n-1)}{w-1}$$

$$= \frac{\left(\text{cis}\left(\frac{2\pi}{n}\right)\right)^n - 1}{\text{cis}\left(\frac{2\pi}{n}\right) - 1}$$

$$= \frac{\text{cis}\, 2\pi - 1}{\text{cis}\left(\frac{2\pi}{n}\right) - 1} \quad \{\text{De Moivre}\}$$

$$= \frac{1-1}{\text{cis}\left(\frac{2\pi}{n}\right) - 1}$$

$$= 0$$

So, $1+w+w^2+....+w^{n-1} = 0$.

page 189 **EXERCISE 4F.2** Question **5**, should read:

5 Let $\alpha = r\,\text{cis}\,\theta$

$\therefore$ $z^n = r\,\text{cis}\,(\theta + k2\pi)$ where $k \in \mathbb{Z}$

$\therefore$ $z = [r\,\text{cis}(\theta+k2\pi)]^{\frac{1}{n}}$

$\therefore$ $z = r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta+k2\pi}{n}\right)$ $\{\text{De Moivre}\}$

$\therefore$ $z = r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}\right)\text{cis}\left(\frac{k2\pi}{n}\right)$

$\therefore$ the n roots of $z^n = \alpha$ are $r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}\right)$, $r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}\right)\text{cis}\left(\frac{2\pi}{n}\right)$, $r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}\right)\text{cis}\left(\frac{4\pi}{n}\right)$, ... ,

$r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta}{n}\right)\text{cis}\left(\frac{2\pi}{n}(n-1)\right)$ $\{\text{letting } k = 0, 1, 2, ..., n-1\}$

$\therefore$ the sum of the n roots of $z^n = \alpha$ is

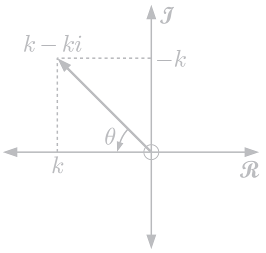
$$r^{\frac{1}{n}}\,\text{cis}\left(\tfrac{\theta}{n}\right) + r^{\frac{1}{n}}\,\text{cis}\left(\tfrac{\theta}{n}\right)\text{cis}\left(\tfrac{2\pi}{n}\right) + r^{\frac{1}{n}}\,\text{cis}\left(\tfrac{\theta}{n}\right)\text{cis}\left(\tfrac{4\pi}{n}\right) + ... + r^{\frac{1}{n}}\,\text{cis}\left(\tfrac{\theta}{n}\right)\text{cis}\left(\tfrac{2\pi}{n}(n-1)\right)$$

$$= r^{\frac{1}{n}}\,\text{cis}\left(\tfrac{\theta}{n}\right)\underbrace{\left[1 + \text{cis}\left(\tfrac{2\pi}{n}\right) + \text{cis}\left(\tfrac{4\pi}{n}\right) + ... + \text{cis}\left(\tfrac{2\pi}{n}(n-1)\right)\right]}_{\text{these are the } n\text{th roots of unity, whose sum } = 0 \quad \{\text{using } \mathbf{4}\}}$$

$$= 0$$

page 199 **REVIEW SET 4B** Question **4 c**, should read:

4 **c**



$$\begin{aligned} |\,k - ki\,| &= \sqrt{k^2 + (-k)^2} \\ &= \sqrt{2k^2} \\ &= |\,k\,|\,\sqrt{2} \end{aligned}$$

Since $k < 0$, $\quad |\,k - ki\,| = -k\sqrt{2}$

$$\begin{aligned} \tan\theta &= \frac{k}{k} \\ &= 1 \qquad \{k \neq 0\} \\ \therefore\ \theta &= \tfrac{\pi}{4} \\ \therefore\ \arg(k - ki) &= \pi - \tfrac{\pi}{4} \\ &= \tfrac{3\pi}{4} \\ \therefore\ k - ki &= -k\sqrt{2}\,\operatorname{cis}\left(\tfrac{3\pi}{4}\right) \end{aligned}$$

which is in polar form since $k < 0$

page 203 **REVIEW SET 4B** Question **17 b**, should read:

17 **a** If $z = \operatorname{cis}\theta$

$$\begin{aligned} &= \cos\theta + i\sin\theta \\ \therefore\ |z| &= \sqrt{\cos^2\theta + \sin^2\theta} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

b If $z = \operatorname{cis}\theta$

then $$\begin{aligned} z^* &= \operatorname{cis}(-\theta) \\ &= (\operatorname{cis}\theta)^{-1} \qquad \{\text{De Moivre's theorem}\} \\ &= z^{-1} \\ &= \frac{1}{z} \end{aligned}$$

The following errata were made on 28/Apr/2020

page 48 **EXERCISE 2C** Question **6 b**, fourth line should read:

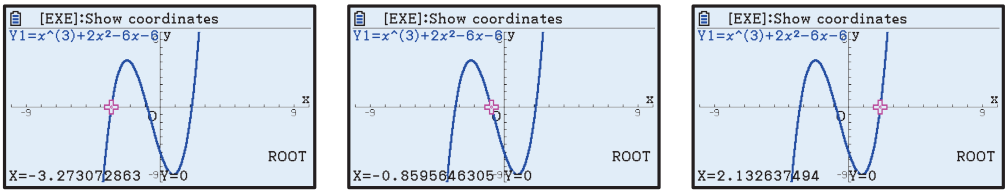
6 **b**

$$\begin{aligned} &3z^3 - z^2 + (a+1)z + a \\ &= (3z+2)(z^2 + bz + c) \quad \text{for some constants } b \text{ and } c \\ &= 3z^3 + 3bz^2 + 3cz + 2z^2 + 2bz + 2c \\ &= 3z^3 + (3b+2)z^2 + (2b+3c)z + 2c \end{aligned}$$

Equating coefficients gives $\begin{cases} 3b + 2 = -1 & \text{.... (1)} \\ 2b + 3c = a + 1 & \text{.... (2)} \\ 2c = a & \text{.... (3)} \end{cases}$

page 83 **EXERCISE 2J** Question **4 a**, should read:

4 **a**

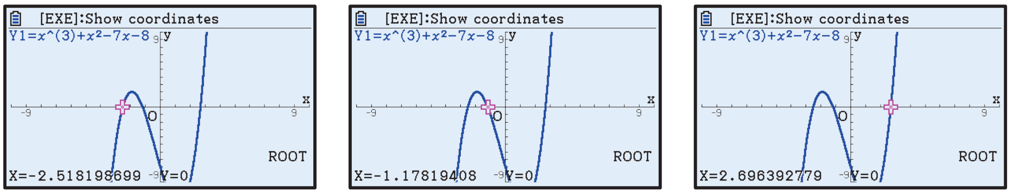


Using technology, $x^3 + 2x^2 - 6x - 6$ has zeros of -3.27, -0.860, and 2.13

$\therefore \quad x \approx -3.27, \ -0.860, \ \text{or} \ 2.13$

page 84 **EXERCISE 2J** Question **4 b**, should read:

4 **b**



Using technology, $x^3 + x^2 - 7x - 8$ has zeros of -2.52, -1.18, and 2.70

$\therefore \quad x \approx -2.52, \ -1.18, \ \text{or} \ 2.70$

page 211 **EXERCISE 5C** Question **2 c**, should read:

2 **a** $2\mathbf{x} = \mathbf{p}$

$\therefore \ \mathbf{x} = \frac{1}{2}\mathbf{p}$

$= \frac{1}{2}\begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -\frac{5}{2} \\ 2 \end{pmatrix}$

b $\frac{1}{3}\mathbf{x} = \mathbf{p}$

$\therefore \ \mathbf{x} = 3\mathbf{p}$

$= 3\begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ -15 \\ 12 \end{pmatrix}$

c $4\mathbf{x} + \mathbf{p} = \mathbf{0}$

$\therefore \ 4\mathbf{x} = -\mathbf{p}$

$\therefore \ \mathbf{x} = -\frac{1}{4}\mathbf{p}$

$= -\frac{1}{4}\begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{4} \\ -1 \end{pmatrix}$

page 242 **EXERCISE 5I.2** Question **3**, should read:

3 If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, then $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\mathbf{0}$

$$\therefore\ \begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}=\mathbf{0}$$

$$\therefore\ (a_2b_3-a_3b_2)\mathbf{i}-(a_1b_3-a_3b_1)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}=\mathbf{0}$$

$$\therefore\ \begin{cases}a_2b_3-a_3b_2=0 & \text{.... (1)}\\ a_1b_3-a_3b_1=0 & \text{.... (2)}\\ a_1b_2-a_2b_1=0 & \text{.... (3)}\end{cases}$$

If $\mathbf{a},\ \mathbf{b}\neq\mathbf{0}$, then at least one component of both **a** and **b** is non-zero. Suppose $a_1,\ b_1\neq 0$.

$$\therefore\ \begin{cases}a_3=\dfrac{a_1}{b_1}\,b_3 & \{(2)\}\\[2ex] a_2=\dfrac{a_1}{b_1}\,b_2 & \{(3)\}\\[2ex] a_1=\dfrac{a_1}{b_1}\,b_1\end{cases}$$

$$\therefore\ \mathbf{a}=\frac{a_1}{b_1}\mathbf{b},\quad \frac{a_1}{b_1}\in\mathbb{R},\quad \frac{a_1}{b_1}\neq 0$$

We can rearrange (1), (2), and (3) similarly for any pair of non-zero components.

$\therefore$ $\mathbf{a}\parallel\mathbf{b}$ for any $\mathbf{a},\ \mathbf{b}\neq\mathbf{0}$

If $\mathbf{a}\parallel\mathbf{b}$ then $\mathbf{b}=k\mathbf{a}$, $k\in\mathbb{R}$, $k\neq 0$

$$\begin{aligned}\therefore\ \mathbf{a}\times\mathbf{b}&=\mathbf{a}\times k\mathbf{a}\\ &=k(\mathbf{a}\times\mathbf{a})\\ &=k\times\mathbf{0}\\ &=\mathbf{0}\end{aligned}$$

$\therefore$ if **a** and **b** are non-zero vectors then $\mathbf{a}\times\mathbf{b}=\mathbf{0}\ \Leftrightarrow\ \mathbf{a}$ is parallel to **b**.

The following erratum was made on 23/Dec/2019

page 154 **EXERCISE 4B** Question **14 a**, second line should read:

14 **a** $z^*=-iz$

$\therefore\ x-iy=-i(x+iy)$

$\therefore\ x-iy=-ix+y$

Equating real and imaginary parts,

$x=y$ and $-y=-x$

$\therefore\ y=x$

The following errata were made on 22/Oct/2019

page 452 **EXERCISE 8H** Question **5 b ii**, should avoid two different uses of the variable name A:

5 b ii

$$\frac{dF}{dt} = kF\left(1 - \frac{F}{95\,000}\right) = kF\left(\frac{95\,000 - F}{95\,000}\right)$$

$$\therefore \quad \frac{95\,000}{F(95\,000 - F)}\frac{dF}{dt} = k$$

$$\therefore \quad \int \frac{95\,000}{F(95\,000 - F)}\,dF = \int k\,dt$$

$$\therefore \quad \int \left(\frac{1}{F} + \frac{1}{95\,000 - F}\right) dF = \int k\,dt$$

$$\therefore \quad \ln|F| - \ln|95\,000 - F| = kt + c$$

$$\therefore \quad \ln\left|\frac{F}{95\,000 - F}\right| = kt + c$$

$$\therefore \quad \frac{F}{95\,000 - F} = \pm e^{kt+c}$$

$$\therefore \quad \frac{95\,000 - F}{F} = Be^{-kt} \qquad \left\{B = \pm\frac{1}{e^c}\right\}$$

Now when $t = 0$, $F = 14$

$$\therefore \quad \frac{95\,000 - 14}{14} = Be^0$$

$$\therefore \quad B = \frac{94\,986}{14}$$

So, we have $\dfrac{95\,000 - F}{F} = \dfrac{94\,986}{14}e^{-kt}$

$$\therefore \quad \frac{95\,000}{F} - 1 = \frac{94\,986}{14}e^{-kt}$$

$$\therefore \quad \frac{95\,000}{F} = 1 + \frac{94\,986}{14}e^{-kt}$$

$$\therefore \quad F = \frac{95\,000}{1 + \frac{94\,986}{14}e^{-kt}}$$

In 1900, $t = 55$ and $F = 30\,000$

$$\therefore \quad 30\,000 = \frac{95\,000}{1 + \frac{94\,986}{14}e^{-55k}}$$

$$\therefore \quad 1 + \frac{94\,986}{14}e^{-55k} = \frac{95\,000}{30\,000}$$

$$\therefore \quad 1 + \frac{94\,986}{14}e^{-55k} = \frac{19}{6}$$

$$\therefore \quad \frac{94\,986}{14}e^{-55k} = \frac{13}{6}$$

$$\therefore \quad e^{-55k} = \frac{91}{284\,958}$$

$$\therefore \quad -55k = \ln\left(\frac{91}{284\,958}\right)$$

$$\therefore \quad k = -\frac{1}{55}\ln\left(\frac{91}{284\,958}\right) \approx 0.146$$

$$\therefore \quad F \approx \frac{95\,000}{1 + \frac{94\,986}{14}e^{-0.146t}}$$

page 489 **EXERCISE 9C** Question **8 d**, should read:

8 **d** Let the speed of the particle be $S(t)$.

$$\begin{aligned} S(t) &= \sqrt{[x'(t)]^2 + [y'(t)]^2} \\ &= \sqrt{(2t-2)^2 + (3t^2-3)^2} \\ &= \sqrt{4t^2 - 8t + 4 + 9t^4 - 18t^2 + 9} \\ &= \sqrt{9t^4 - 14t^2 - 8t + 13} \end{aligned}$$

$$\therefore \quad [S(t)]^2 = 9t^4 - 14t^2 - 8t + 13$$

$$\begin{aligned} \frac{d}{dt}[S(t)]^2 &= 36t^3 - 28t - 8 \\ &= 0 \text{ when } t = 1 \qquad \{\text{technology, } t > 0\} \end{aligned}$$

The maximum and minimum speeds occur either at $t = 1$ or at the boundaries.

When $t = 0$, $\mathbf{v} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$\therefore$ speed $= \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$ cm s^{-1}

When $t = 5$, $\mathbf{v} = \begin{pmatrix} 8 \\ 72 \end{pmatrix}$

$\therefore$ speed $= \sqrt{8^2 + 72^2} = \sqrt{5248}$ cm s^{-1}

t	0	1	5
speed	$\sqrt{13}$	0	$\sqrt{5248}$

$\therefore$ minimum speed $= 0$ cm s^{-1},
maximum speed $= \sqrt{5248} \approx 72.4$ cm s^{-1}

The following erratum was made on 16/Sep/2019

page 434 **EXERCISE 8E** Question **4 e**, should read:

4 **e** $e^y(2x^2 + 4x + 1)\dfrac{dy}{dx} = (x+1)(e^y + 3)$

$$\therefore \quad \frac{e^y}{e^y + 3}\frac{dy}{dx} = \frac{x+1}{2x^2 + 4x + 1}$$

$$\therefore \quad \int \frac{e^y}{e^y+3}\frac{dy}{dx}\,dx = \int \frac{x+1}{2x^2+4x+1}\,dx$$

$$\therefore \quad \int \frac{e^y}{e^y+3}\,dy = \tfrac{1}{4}\int \frac{4x+4}{2x^2+4x+1}\,dx$$

$$\therefore \quad \ln|e^y + 3| = \tfrac{1}{4}\ln|2x^2 + 4x + 1| + c$$

$$\therefore \quad e^y + 3 = A\left|2x^2 + 4x + 1\right|^{\frac{1}{4}} \qquad \{A = \pm e^c, \ e^y + 3 > 0 \text{ for } y \in \mathbb{R}\}$$

$$\therefore \quad e^y = A\left|2x^2 + 4x + 1\right|^{\frac{1}{4}} - 3$$

$$\therefore \quad y = \ln\left[A\left|2x^2 + 4x + 1\right|^{\frac{1}{4}} - 3\right]$$

But $y(0) = 2$, so $2 = \ln(A - 3)$

$\therefore \quad e^2 = A - 3$

$\therefore \quad A = e^2 + 3$

The particular solution is $y = \ln\left[\sqrt[4]{|2x^2 + 4x + 1|}\,(e^2 + 3) - 3\right]$.

The following erratum was made on 02/Sep/2019

page 425 **EXERCISE 8C** Question **4 c**, should read:

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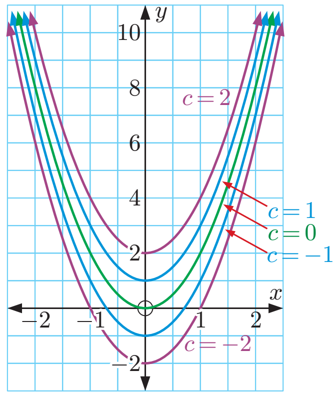
4 **a** If $y = 2x^2 + c$, then $\dfrac{dy}{dx} = 4x$ for any constant c as required.

b



c From **a**, $y = 2x^2 + c$ is a general solution to the differential equation.

The particular solution passes through $\left(1, \frac{1}{2}\right)$, so

$$\tfrac{1}{2} = 2(1)^2 + c$$

$\therefore\ c = \frac{1}{2} - 2 = -\frac{3}{2}$

$\therefore$ the particular solution is $y = 2x^2 - \frac{3}{2}$

The following erratum was made on 14/Aug/2019

page 415 **EXERCISE 8B** Question **4**, should state correct units:

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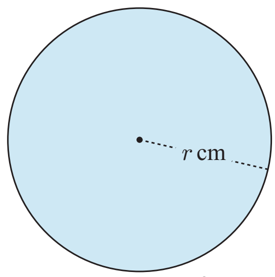
4 The area of the circular ripple is $A = \pi r^2$.

Differentiating both sides of $A = \pi r^2$ with respect to t:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Since the ripple moves out at a constant speed of $1\ \mathrm{m\,s^{-1}}$,

$\dfrac{dr}{dt} = 1\ \mathrm{m\,s^{-1}}$.



Area $= A\ \mathrm{cm}^2$

a When $r = 2$, $\dfrac{dA}{dt} = 2\pi \times 2 \times 1 = 4\pi\ \mathrm{m^2\,s^{-1}}$

$\therefore$ the circle's area is increasing at $4\pi\ \mathrm{m}^2$ per second.

b When $r = 4$, $\dfrac{dA}{dt} = 2\pi \times 4 \times 1 = 8\pi\ \mathrm{m^2\,s^{-1}}$

$\therefore$ the circle's area is increasing at $8\pi\ \mathrm{m}^2$ per second.

The following errata were made on 12/Jul/2019

page 89 **REVIEW SET 2A** Question **11**, last line should read:

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11 $= a(z^4 - 6z^3 + 14z^2 - 10z - 7), \quad a \in \mathbb{Q}, \quad a \neq 0$

page 178 **EXERCISE 4E** Question **13**, last two lines should read:

---

13 **c** **i** $\therefore\ \left| z^1 \times z^2 \times z^3 \times \ldots. \times z^6 \right| = 2^{\frac{6(7)}{2}}$

$= 2^{21}$

page 284 **EXERCISE 6G.2** Question **1 c**, should read:

1 c Line 1 has direction vector $\begin{pmatrix}6\\8\\2\end{pmatrix}$ and line 2 has direction vector $\begin{pmatrix}3\\4\\1\end{pmatrix}$.

As $\begin{pmatrix}6\\8\\2\end{pmatrix} = 2\begin{pmatrix}3\\4\\1\end{pmatrix}$ the two lines are parallel. Hence, $\theta = 0°$.

To see if the lines are coincident, try to find a shared point.

When $t = 0$, the point on line 1 is $(0,\ 3,\ -1)$.

$\therefore$ the unique point on line 2 with z-coordinate -1 is the point where $1 + s = -1$

$\therefore$ $s = -2$.

This point is $(-4,\ -8,\ -1)$.

Since $(0,\ 3,\ -1) \neq (-4,\ -8,\ -1)$, the lines are not coincident.

page 417 **EXERCISE 8B** Question **10 b**, should read:

10 b When $x = 10$, $\dfrac{dS}{dt} = -\dfrac{80}{(40-10)^2} = -\frac{80}{900} = -\frac{4}{45}$ m s^{-1}

$\therefore$ the person's shadow is shortening at $\frac{4}{45}$ m s^{-1}

The following errata were made on or before 10/Jul/2019

page 108 **EXERCISE 3B** Question **8**, last line should read:

8 $\therefore$ if $f(x)$ is never increasing, then $-1 \leqslant k \leqslant 0$.

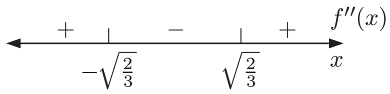
page 125 **EXERCISE 3D.1** Question **4 g ii**, sign diagram should be:

4 g ii $f''(x) = 12x^2 - 8$

$= 4(3x^2 - 2)$

$= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$

$\therefore$ $f''(x)$ has sign diagram:



page 158 **EXERCISE 3G** Question **10**, change the lines before **Sign diagram test** to:

10 ....

$\therefore$ $-3x^2 + 40x - 100 = 0$

$\therefore$ $-(3x - 10)(x - 10) = 0$

$\therefore$ $x = \frac{10}{3} = 3\frac{1}{3}$ $\{$as $0 \leqslant x \leqslant 5\}$

....

page 169 **REVIEW SET 3A** Question **11 b ii**, last line should read:

11 b ii $\therefore$ it will take ≈ 201 years for the tree to reach a height of 300 cm.

page 201 **EXERCISE 4E.2** Question **2 d**, last line should read:

2 d $\therefore$ $f(x) = e^x + 3\sin x - e^{\pi}$

page 229 **EXERCISE 5B** Question **9 b**, diagram should have region filled for $0 \leqslant x \leqslant 2$:

9 b $f(x) = -x(x-2)(x-4)$

$\therefore$ $y = f(x)$ cuts the x-axis at 0, 2, and 4.

$f(x) = -x(x-2)(x-4)$

$= -x(x^2 - 6x + 8)$

$= -x^3 + 6x^2 - 8x$